# Assignment 7

Hand in no. 2, 4, 5, and 11 by April 4.

1. Consider the linear partial differential equation of second order with two variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G\ ,$$

where $A, B, C, D, E, F$ and $G$ are given functions of $(x, y)$ in some plane region $D$. The equation is called homogeneous if $G \equiv 0$. Show that if $u_1$ and $u_2$ are solutions to this equation in the homogeneous case, $a_1u_1 + a_2u_2$ is again a solution for any $a_1$ and $a_2$. When $G \neq 0$, show that every solution can be written as $u = v + w$ where $v$ is a solution to the corresponding homogeneous equation and $w$ is a particular solution to the full equation. (Note. The same result applies to all linear PDE's of all orders and variables.)

2. Consider the initial-boundary value problem under the Neumann condition

$$\begin{cases} u_t = u_{xx} & \text{in } [0,\pi]\times(0,\infty)\ , \\ u(x,0) = f(x) & \text{in } [0,\pi], \\ u_x(0,t) = u_x(\pi,t) = 0\ , \quad t>0, & \end{cases} \tag{1}$$

   (a) By extending the solution to $[-\pi, \pi]$ as an even function in $x$, use cosine series to find the solution of this problem. (This was done in class.)

   (b) Use the method of separation of variables to solve the problem.

3. Optional. Instead of separation of variables, use Fourier series to study the normalized heat equation under $u_x(0,t) = 0,\ u(\pi,t) = 0$ . Hint: You need to extend $u$ to become a $4\pi$-periodic function.

4. Consider the heat equation ($l = 1, \kappa = 1$) under the Robin condition $u(0,t) = 0,\ u_x(1,t)+u(1,t) = 0$. Show that all eigenvalues of the corresponding problem are given by $\lambda_n, n \geq 0$, where $\lambda_n \in ((2n-1)^2\pi^2/4, n^2\pi^2)$ either analytically or by plotting graphs. Then find a formal solution to this problem.

5. Consider the eigenvalue problem on $[0,1]$:

$$(p(x)X')' + q(x)X = -\lambda X, \quad \alpha X'(0) + \beta X(0) = 0, \quad \gamma X'(1) + \delta X(1) = 0\ ,$$

where $p, q$ are nice functions and $\alpha\beta \neq 0,\ \gamma\delta \neq 0$. Show that the eigenfunctions corresponding to different eigenvalues are orthogonal on $[0,1]$.

6. Find all solutions to the first order equation $u_t = cu_x$ where $c$ is a non-zero constant of the form $X(x)T(t)$. Can you use them to solve the initial-boundary value problem for this equation under Dirichlet boundary condition?

7. In (5), Ex 6, we solve the initial-boundary value problem for the wave equation. Show that the solution $u$ can be expressed in the following close form:

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy \ .$$

8. Verify that the general solution to the ordinary differential equation $x'' + bx' + ax = 0$, $a, b \in \mathbb{R}, b^2 \neq 4a$, is given by $x(t) = Ae^{\alpha t} + Be^{\beta t}$ where $\alpha, \beta \in \mathbb{C}$, are the roots of the quadratic equation $y^2 + by + a = 0$. Can you find the general solution when $b^2 = 4a$ ?

9. Consider the modified wave equation $u_{tt} = u_{xx} - 2\beta u_t$, $\beta \in (0,1)$, under the Dirichlet condition $u(0,t) = u(\pi,t) = 0$ and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$. Using Fourier series or separation of variables to show that the formal solution is given by

$$u(x,t) = e^{-\beta t}\sum_{n=1}^{\infty} B_n\left(\cos\alpha_n t + \frac{\beta}{\alpha_n}\sin\alpha_n t\right)\sin nx \ , \quad \alpha_n = \sqrt{n^2-\beta^2} \ ,$$

where $B_n$ is the coefficient of the sine series of $f$. Hint: You need the previous problem.

10. Consider the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} - g \ ,$$

where $g$ is a positive constant under the Dirchlet condition $u(0,t) = u(\pi,t) = 0$ and initial conditions $u(x,0) = u_t(x,0) = 0$. Use separation of variables to show that a formal solution is given by

$$u(x,t) = \frac{4g}{\pi c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos(2n-1)ct\ \sin(2n-1)x - \frac{g}{2c^2}x(\pi - x) \ .$$

11. Consider

$$\begin{cases} u_t = \kappa u_{xx} + F(x,t) & (x,t) \in [0,\pi]\times[0,\infty),\ \kappa > 0, \\ u(x,0) = f(x) \ , & x \in [0,\pi], \\ u_x(0,t) - au(0,t) = \phi(t), & \\ u_x(\pi,t) + bu(\pi,t) = \psi(t), \quad a,b > 0, \quad t > 0 & \end{cases}$$

This is a nonhomogeneous heat equation with nonhomogeneous boundary conditions. Show that it has at most one solution. You may proceed formally by assuming the solutions are as regular as possible. Let $w = u_2 - u_1$ where $u_1$ and $u_2$ are solutions and show $w \equiv 0$. Suggestion: Differentiate the integral

$$\int_0^{\pi} w^2(x,t)dx$$

in time.

12. Find the general solution to the ordinary differential equation

$$t^2x''(t) + tx'(t) - a^2x(t) = 0, \quad a > 0 \ .$$

Hint: Look at the equation satisfied by $y(r) = x(t),\ t = e^r$ . The answer is

$$x(t) = At^a + Bt^{-a}, \quad A, B \in \mathbb{R} \ .$$

13. Consider

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 & \text{in } \Omega, \\ u = \varphi \text{ on } \partial\Omega \ , \end{cases}$$

where $\Omega$ is the plane domain bounded by the arcs $r = 1, r = 2$ and $\theta = 0, \theta = \pi$. The boundary data $\varphi$ is zero on $r = 1, \theta = 0, \pi$ and equal to a constant $c_0$ on $r = 2$. Roughly speaking, this domain is half of the region bounded by two concentric circles $r = 2$ and $r = 1$. Use separation of variables to solve this problem. Hint: The previous problem is needed.

14. Optional. Prove the formula

$$1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1 - r^2}{1 - 2r\cos x + r^2} \ , \quad r \in [0, 1).$$

15. Consider the two dimensional Laplace equation on the rectangle $R = \{(x, y) : x \in [0, l], y \in [0, L]\}$ satisfying the boundary conditions $u(0, y) = y(l, y) = 0, u(x, 0) = f_1(x), u(x, L) = f_2(x)$ . Using separation of variables to show that the formal solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(\alpha_n \cosh\frac{n\pi y}{l} + \beta_n \sinh\frac{n\pi y}{l}\right) \ ,$$

where $\alpha_n$ and $\beta_n$ are determined from the relation

$$\alpha_n = a_n, \quad \alpha_n \cosh\frac{n\pi L}{l} + \beta_n \sinh\frac{n\pi L}{l} = b_n \ ,$$

and $a_n, b_n$ are defined via

$$f_1(x) \sim \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \ , \quad f_2(y) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \ .$$

Here $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$.